GEOMETRY OF DOMAINS WITH THE UNIFORM SQUEEZING PROPERTY

SAI-KEE YEUNG

Abstract *We introduce the notion of domains with uniform squeezing property, study various analytic and geometric properties of such domains and show that they cover many interesting examples, including Teichmüller spaces and Hermitian symmetric spaces of non-compact type. The properties supported by such manifolds include pseudoconvexity, hyperconvexity, Kähler-hyperbolicity, vanishing of cohomology groups and quasi-isometry of various invariant metrics. It also leads to nice geometric properties for manifolds covered by bounded domains and a simple criterion to provide positive examples to a problem of Serre about Stein properties of holomorphic fiber bundles.*

§0 Introduction

The purpose of this article is to introduce a class of bounded domains in $\mathbb{C}^n$ which on one hand is sufficiently general to include interesting classes of examples and on the other hand leads to interesting analytic and geometric properties.

Definition 1. *Denote by $B_r(x)$ a ball of radius r in $\mathbb{C}^n$. Let $0 < a < b < \infty$ be positive constants. A bounded domain M in $\mathbb{C}^n$ for some $n > 1$ is said to have the uniform squeezing property, or more precisely, (a, b)-uniform squeezing property if there exist constants a and b, such that for each point $x \in M$, there exists an embedding $\varphi_x : M \to \mathbb{C}^n$ with $\varphi_x(x) = 0$ and $B_a(\varphi_x(x)) \subset \varphi_x(M) \subset B_b(\varphi_x(x))$. We call the corresponding coordinate system a uniform squeezing coordinate system or, more precisely, (a, b)-uniform coordinate system.*

Even though the definition is very simple and appears to be rather restrictive, it in fact includes lots of interesting examples.

Proposition 1. *Examples of bounded domains with the uniform squeezing property include the followings,*
(a). bounded homogeneous domains,
(b). bounded strongly convex domains,
(c). bounded domains which cover a compact Kähler manifold, and
(d). Teichmüller spaces $\mathcal{T}_{g,n}$ of hyperbolic Riemann surfaces of genus g with n punctures.

Key Words: bounded domains, pseudoconvexity, Kähler-Einstein metric, Kähler hyperbolic, moduli space
1991 Mathematics Subject Classification. Primary 32G15, 53C55, 55N99
The author was partially supported by grants from the National Science Foundation and the National Security Agency.

We remark that Hermitian symmetric spaces of non-compact type constitute an important subclass of both (a) and (c). The former follows from the Harish-Chandra realization of such symmetric spaces, and the latter follows from existence of cocompact arithemtic lattices associated to the automorphism groups of the symmetric spaces (cf. [B]).

Our main objective is to show that domains with uniform squeezing properties support many interesting geometric and analytic properties. The first observation is about pseudoconvexity of such domains.

Theorem 1. *Let M be a domain with the uniform squeezing property. Then the following conclusions are valid.*
(a). The Bergman metric of M is complete.
(b). M is a pseudoconvex domain.
(c). There exists a complete Kähler-Einstein metric on M.

The second observation is about the behavior of invariant metrics on such domains. On a general bounded domain, there are three well-known intrinsic metrics which are invariant under a biholomorphism, namely, the Kobayashi metric, the Carathéodory metric and the Bergman metric. There is a fourth one when the bounded domain is pseudoconvex, viz., the Kähler-Einstein metric. We denote the metrics by g_K, g_C, g_B and g_{KE} respectively. For a Kähler metric g on M, we denote by R^g its curvature tensor and ∇^g the Riemannian connection. For tangent vectors $X_1, \cdots, X_N$, we denote $\nabla^g_{X_1} \cdots \nabla^g_{X_N}$ by $\nabla^g_{X_1,\cdots X_N}$. Furthermore, $\nabla^g_{i_1,\cdots i_N}$ denotes the covariant derivatives with respect to the coordinate vectors. We also normalized the Kähler-Einstein metric so that $Ric(g_{KE}) = -2(n+1)$.

Theorem 2. *Let M be a bounded domain with (a,b)-uniform squeezing property.*
(a). The invariant metrics g_K, g_C, g_B and g_{KE} are quasi-isometric. Furthermore

$$\begin{aligned}
\frac{a}{b} g_K \leqslant \quad & g_C \quad \leqslant g_K \\
\frac{a}{b} g_K \leqslant \quad & g_B \quad \leqslant [\frac{2\pi}{a^3}(\frac{2b}{a})^n]^2 g_K \\
\frac{a^2}{b^2 n} g_K \leqslant \quad & g_{KE} \quad \leqslant (\frac{b^{4n-2} n^{n-1}}{a}) g_K
\end{aligned}$$

(b). There exist constants $c_n^{g_B}$ and $c_n^{g_{KE}}$ such that $\|\nabla^{g_B}_{i_1,\cdots i_N} R_{g_B}\|_{g_B} \leqslant c_N^{g_B}$ and $\|\nabla^{g_{KE}}_{i_1,\cdots i_N} R_{g_{KE}}\|_{g_{KE}} \leqslant c_N^{g_{KE}}$ for any covariant derivatives $\nabla^{g_B}_{i_1,\cdots i_N}$ and $\nabla^{g_{KE}}_{i_1,\cdots i_n}$ of g_B and g_{KE} respectively.
(c). Let $X_1, \cdots, X_N$ be N tangent vectors of unit length with respect to a metric g_1 at $x \in M$. Then $\|\nabla^{g_1}_{X_1,\cdots X_N} g_B - \nabla^{g_1}_{X_1,\cdots X_N} g_{KE}\|_{g_1} \leqslant c$, for some constant c depending on N, where $g_1 = g_B$ or g_{KE}.
(d). Both of g_B and g_{KE} are geometrically finite in the sense that they are complete with bounded curvature and the injectivity radius is bounded from below uniformly on M.
(e). Both of g_B and g_{KE} are Kähler-hyperbolic.
(f). M is hyperconvex.

Here we recall that a Kähler manifold (X, ω) is Kähler-hyperbolic if on its universal covering, ω can be written as dh where h is bounded uniformly when measured with respect to ω. M is hyperconvex if there exists a plurisubharmonic exhaustion function bounded from above on M.

The followings are some well-known consequences of Kähler-hyperbolicity in Theorem 1 and Theorem 2.

Corollary 1. *Let M be a uniformly squeezed manifold. Let g be g_B or g_{KE}.*
(a) The reduced L^2-cohomology groups of M with respect to $g = g_B$ or g_{KE} satisfies $\dim(H^0_{(2)}(M)) = \infty$ and $\dim(H^i_{(2)}(M)) = 0$ for all $i > 0$.
(b) The first eigenvalues of the Beltrami Laplacian operators Δ_{g_B} and $\Delta_{g_{KE}}$ on smooth functions on M with respect to g_B and g_{KE} respectively are both bounded from below by 0.
(c) The volume with respect to either $g = g_B$ or g_{KE} of any relatively compact complex submanifold with boundary $N \subset M$ of complex dimension k satisfies $vol_{2k}(N) \leqslant C \cdot vol_{2k-1}(\partial N)$ for some constant $C > 0$.

We define a lattice on M to be a discrete group acting properly discontinuously as biholomorphisms on M.

Corollary 2. *Assume that Γ is a torsion-free lattice on M which admits a uniform squeezing coordinate system. Then a compact quotient $N = M/\Gamma$ has to be a projective algebraic variety of general type. A non-compact quotient $N = M/\Gamma$ which has finite volume with respect to the invariant Bergman metric has to be a quasi-projective variety of log-general type.*

Another direct consequence of Theorem 2b and a result of Mok-Yau in [MY] is the following estimates on the growth of Bergman kernel.

Corollary 3. *Let M be a bounded domain with the uniform squeezing property. Denote by $d = d(z, \partial\Omega)$ the Euclidean distance of $z \in \Omega$ from the boundary $\partial\Omega$ of Ω. Then $K(z, z) \geqslant \frac{c}{d^2(-\log d)^2}$ for some constant $c > 0$.*

Let us now focus on the applications of the above results to more specific compact or non-compact manifolds.

Theorem 3. *Assume that N is a compact complex manifold of complex dimension n whose universal covering is a bounded domain in $\mathbb{C}^n$. Then the following properties hold.*
(a). N is projective algebraic.
(b). There exists a Kähler-Einstein metric on N.
(c). $(-1)^n\chi(N) > 0$.
(d). $H^0(N, 2K)$ is non-trivial, where K is the canonical line bundle on N.
(e). The universal covering of N is Stein.

The result can be considered as a support for a conjecture of Shafarevich, which states that the universal covering of a projective algebraic variety is holomorphically convex (cf. [Ko]). The assumption is stronger but the projective algebraicity is obtained for free. On the other hand, it also shows that if we try to formulate a conjecture for the uniformization of a compact complex manifold by a bounded domain, it should include topological and analytic conditions such as those stated in (c) and (d). Properties in (c) is along the line of conjectures of Hopf, Chern and Singer in Riemannian geometry, a consequence of those is that the Euler characteristic of a compact Riemannian manifold of even dimension $2n$ with non-positive Riemannian sectional curvature satisfies (c) (cf. [Gr]). Note that a compact torus is flat and its Euler characteristic is equal to zero.

As a consequence of a result of Stehlé, Theorem 2f provides the following simple criterion for positive solutions to a problem of Serre [Se], who asked whether a holomorphic fiber bundle with Stein base and Stein fibers are Stein.

Corollary 4. *Suppose $\pi : T \to B$ is a locally trivial holomorphic fiber space for which the base B is a Stein space and the fibers satisfy the uniform squeezing properties. Then T is also Stein.*

As an application of Theorem 1 and 2 to non-compact complex manifolds of finite volume with respect to some invariant metric, we consider moduli space of possibly punctured curves as an example.

Theorem 4. *Let $g, n \geqslant 0$ and $2g-2+n>0$, so that the complement of n punctures of a compact Riemann surface of genus g gives a hyperbolic Riemann surface. Let $\mathcal{M}_{g,n}$ be the moduli space of such hyperbolic Riemann surfaces. Let $\mathcal{T}_{g,n}$ be the corresponding Teichmüller space.*
(a). g_K, g_C, g_B and g_{KE} are quasi-isometric on $\mathcal{M}_{g,n}$
(b). For g_B and g_{KE}, any order of covariant derivative of the curvature tensor of the metric is uniformly bounded on $\mathcal{M}_{g,n}$. As a consequence, for any set of unit vectors $\{X_1, \cdots X_N\}$ measured with respect to g_1, the difference $\|\nabla^{g_1}_{X_1,\cdots X_N} R^{g_1} - \nabla^{g_2}_{X_1,\cdots X_N} R^{g_2}\|_{g_1}$ is bounded for any g_1, g_2 chosen among g_B, g_{KE}, where $\nabla^{g}_{X_1,\cdots X_N}$ denotes the covariant derivatives of a metric g with respect to the vectors $X_1, \cdots, X_n$.
(c). The Teichmüller space $\mathcal{T}_{g,n}$ is Kähler-hyperbolic with respect to both g_B and g_{KE}.
(d). $\mathcal{T}_{g,n}$ is hyper-convex.
(f). $\mathcal{M}_{g,n}$ is quasi-projective of log-general type and the Euler-Poincaré characteristic satisfies $(-1)^n\chi(\mathcal{M}_{g,n}) > 0$.

Except for the statements related to the Kähler-hyperbolicity of the Kähler-Einstein metric and estimates on higher order quasi-isometry of the metrics g_B and g_{KE}, most of the results in Theorem 4 can be obtained for example by combining results in [Y3] and [Y4], but the proofs there rely on many well-known and diverse results. In this paper, all these properties except quasi-projectivity of moduli space of curves are derived solely from the existence of uniform squeezing coordinates, which is provided classically by the Bers Embedding (cf. [Ga]).

Overall we remark that parts of the results in this paper have been obtained for some specific examples mentioned in Proposition 1. In particular, Kähler-hyperbolicity of locally Hermitian symmetric spaces with respect to the Bergman metric is explained in [Gr], of bounded homogeneous space with respect to the Bergman metric is proved in [Do], of moduli space of curves with respect to a metric constructed by McMullen is proved by McMullen in [Mc], where the metric is also shown to have many nice properties such as geometric finiteness and quasi-isometry to the Kobayashi metric. The vanishing of the cohomology groups $h^i_{(2)}, i < n$, for Kähler-hyperbolic manifolds was proved by Gromov in [Gr].

There is a vast amount of literature related to Theorem 4. A precise formula for the Euler characteristic of the moduli space of curves was given by Harer-Zagier. $\mathcal{M}_{g,n}$ was first shown to be pseudoconvex or a domain of holomorphy through the work of Bers and Ehrenpreis. Hyperconvexity of $\mathcal{M}_{g,n}$ had been proved by Krushkal, and more recently in [Y2] by a different and more geometric way

using results of Wolpert. It follows from classical results of Baily, Deligne-Mumford and Knudson-Mumford that a moduli space of curves is quasi-projective. Quasi-isometry among invariant metrics on $\mathcal{M}_{g,n}$ were obtained through the contributions of Chen, Liu-Sun-Yau and Yeung. We refer the readers to [W] and [Y4] for more details.

In a sequel to the present paper, we will explain how the set-up of the paper can be used to prove subelliptic estimates for solutions to the $\overline{\partial}$ equations on uniform squeezing domains, which include Teichmüller spaces, where the problem remained open in the past due to a lack of description of the boundary of Bers Embedding.

Parts of the work were finished while the author was visiting the Korea Institute for Advanced Study, the Osaka University, and the University of Hong Kong. The author would like to thank Jun-Muk Hwang, Toshiki Mabuchi and Ngaiming Mok for their hospitality. The author would also like to express his gratitude to the referee for very helpful comments on the paper.

§1 Terminology and preliminaries

Recall the following standard notations about various convexity of a domain. A domain in $\mathbb{C}^n$ is pseudoconvex if there exists a plurisubharmonic exhaustion. A bounded domain $\Omega = \{z \in \mathbb{C}^n | r(z) < 0\}$ for some C^2 function $r(z)$ in $z = (z_1, \cdots, z_n)$ is strongly pseudoconvex if the Levi form $\sqrt{-1}\partial\overline{\partial}r > 0$ in a neighborhood of $\partial\Omega$. A domain Ω in $\mathbb{C}^n$ is hyperconvex if there exists a bounded plurisubharmonic exhaustion function.

A Kähler metric ω on a complex manifold M is said to be Kähler-hyperbolic if on the universal covering $\tilde{M}$ of M, the pull back of ω can be expressed as $d\eta$ for some 1-form η which is bounded uniformly on $\tilde{M}$ with respect to ω.

We say that two metrics g_1 and g_2 are quasi-isometric, denoted by $g_1 \sim g_2$, if there exists a positive constant c such that $\frac{1}{c} g_1(v, \overline{v}) \leqslant g_2(v, \overline{v}) \leqslant c g_1(v, \overline{v})$ for all holomorphic tangent vectors v.

Let us now recall the various notions of invariant metrics on a general complex manifold.

For a unit tangent vector $v \in T_x M$ on a complex manifold M, the Kobayashi and Carathéodory semi-metrics are defined respectively as complex Finsler metrics by

$$
\begin{aligned}
\sqrt{g_K(x,v)} &= \inf\{\frac{1}{r} | \exists f : B_r^1 \to M \text{ holomorphic}, f(0) = x, f'(0) = v\}. \\
\sqrt{g_C(x,v)} &= \sup\{\frac{1}{r} | \exists h : M \to B_r^1 \text{ holomorphic}, h(x) = 0, |dh(v)| = 1\},
\end{aligned}
$$

where we use $B_r^n = B_r^n(0)$ to denote a ball of radius r centered at 0 in $\mathbb{C}^n$. Since we are considering only bounded domains in $\mathbb{C}^n$, both g_K and g_C are non-degenerate complex Finsler metrics.

Consider now Kähler-Einstein metric of constant negative scalar curvature. We normalize the curvature so that g_{KE} satisfies $Ric(g_{KE}) = -2(n+1)$, where ω_{KE} is the Kähler form associated to g_{KE}. The normalization is chosen so that it agrees with the one for the hyperbolic metric on $B_{\mathbb{C}}^n$ of constant holomorphic sectional curvature -4.

The Bergman pseudometric g_B on a general complex manifold M of complex dimension N is a Kähler pseudometric with local potential given by the coefficients of the Bergman kernel $K(x,x)$. It is clearly non-degenerate for $\mathcal{T}_g$. g_B can be interpreted in the following way.

Let f be a L^2-holomorphic N-form on M, where $\dim_{\mathbb{C}} M = N$. In terms of local coordinates $(z_1, \cdots, z_N)$ on a coordinate chart U, let $e_{K_M} = dz^1 \wedge \cdots \wedge dz^N$ be a local basis of the canonical line bundle K_M on U. We can write f as $f_U e_{K_M}$ on U. Let $f_i, i \in N$ be an orthonormal basis of L^2-sections in $H^0_{(2)}(M, K_M)$. Note that from conformality, the choice is independent of the metric on M. The Bergman kernel is given by $K(x,x) = \sum_i f_i \wedge \overline{f_i}$. Let $K_U(x,x) = \sum_i f_{U,i}\overline{f_{U,i}}$ be the coefficient of $K(x,x)$ in terms of the local coordinates. The Bergman metric is given by a Kähler form

$$\omega_B = \sqrt{-1}\partial\overline{\partial}\log K_U(x,x) = \sqrt{-1}\frac{1}{K_U(x,x)^2}\sum_{i<j}(f_i\partial f_j - f_j\partial f_i)\wedge\overline{(f_i\partial f_j - f_j\partial f_i)},$$

which is clearly independent of the choice of a basis and U. As the Bergman kernel is independent of basis, for each fixed point $x \in M$,

$$K_U(x,x) = \sup_{f\in H^0_{(2)}(M,K_M), \|f\|=1} |f_U(x)|^2,$$

where $\|\cdot\|$ stands for the L^2-norm. We may assume that $\sup_{f\in H^0_{(2)}(M,K_M)} |f_U(x)|$ is realized by $f_x \in H^0_{(2)}(M,K)$ with $\|f_x\| = 1$ so that $K_U(x,x) = |f_{x,U}(x)|^2$. Using the fact that the Bergman kernel is independent of the choice of a basis again and letting $V \in T_xM$,

$$\omega_B(V,\bar{V}) = \frac{1}{|f_{x,U}(x)|^2}\sup_{f\in H^0_{(2)}(M,K_M), \|f\|=1, f(x)=0} |V(f_U)|^2.$$

Consider in particular $V = \frac{\partial}{\partial z^i}$. We may also assume that the supremum for $|\frac{\partial}{\partial z^i}(f_U)|^2$ among all $f \in H^0_{(2)}(M,K_M), \|f\| = 1, f(x) = 0$ is achieved by $g_{i,x} \in H^0_{(2)}(M,K_M)$ of L^2-norm 1. Hence $\sup_{f\in H^0_{(2)}(M,K_M),\|f\|=1} |\frac{\partial}{\partial z^i} f_U|^2 = |\frac{\partial}{\partial z^i} g_{i,x,U}(x)|^2$. To simplify our notation, we may simply write

$$\omega_B(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^i}}) = \frac{1}{|f_x(x)|^2}\sup_{f\in H^0_{(2)}(M,K_M), \|f\|=1, f(0)=0} |\frac{\partial}{\partial z^i}(f)|^2 = \frac{|\frac{\partial}{\partial z^i} g_{i,x}(x)|^2}{|f_x(x)|^2},$$

since the expression is clearly independent of the choice of U and metric on e_U.

Finally let us include here two regularity estimates required for later calculations for the convenience of the readers. We denote by $W_{k,p}$ and $C_{k,\alpha}$ the spaces of functions on B_a which are bounded with respect to the Sobolev norm $\|\cdot\|_{k,p}$ and Hölder norm $|\cdot|_{k,\alpha}$ on $B_a(x)$ respectively. We refer the readers to [GT] for standard notations.

Proposition 2. *(cf.[GT], page 235, 90) Let $\Omega' \subset\subset \Omega$ be bounded domains in $\mathbb{R}^n$ with C^∞ boundary. Let L be a second order linear differential operator defined by*

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$$

with sums over repeated indices. Let u be a strong solution of the equation $Lu = f$. (a) (Calderon-Zygmund estimates) Suppose that $u \in H_1(\Omega)$ is a strong solution of

$Lu = f$ with $f \in L^2(\Omega)$. Assume that for $a^{ij}, b^i, c \in C^0(\bar{\Omega})$,

$$\begin{aligned} a^{ij} v_i v_j &\geq \lambda |v|^2 \quad \forall v \in \mathbb{R}^n, \\ |a^{ij}|, |b^i|, |c| &\leqslant \Lambda. \end{aligned}$$

Then

$$\|u\|_{2,p,\Omega'} \leqslant C_1(\|u\|_{0,p,\Omega} + \|f\|_{0,p,\Omega})$$

with constant C_1 depending on $n, p, \lambda, \Lambda, \Omega', \Omega$ and the moduli of continuity of a^{ij} on Ω'.
(b) (Schauder estimates) Suppose $f \in C^\alpha(\bar{\Omega})$ and

$$|a^{ij}|_{\alpha,\Omega}, |b^i|_{\alpha,\Omega}, |c|_{\alpha,\Omega} \leqslant \Lambda.$$

Then

$$|u|_{2,\alpha,\Omega'} \leqslant C_2(|u|_{0,\alpha} + |f|_{0,\alpha,\Omega}),$$

with constant C_2 depending on $n, \alpha, \lambda, \Lambda, \Omega$ and Ω'.

In our application, we will always assume that $\Omega = B_a(0)$ and $\Omega' = B_{a/2}(0)$ for a fixed value a, after identifying an arbitrary point on the manifold to the origin with respect to a uniform squeezing coordinate system. We are interested in the estimates of the bounds and use it to show uniform bound over the manifold of our interest instead of regularity, which is already known for general elliptic equations.

§2 Pseudoconvexity and related properties

Throughout this section and §3, §4, we let M be a bounded domain with uniform squeezing coordinates.

Lemma 1. *The Bergman metric g_B on M is a well-defined complete Kähler metric. Furthermore, g_B is quasi-isometric to g_K as a complex Finsler metric.*

Proof The $(1,1)$-form ω_B defined in §1 is only semi-definite in general. We need to show that it is in fact positive definite and gives rise to a complete metric in our situation.

The Kähler form ω of the Bergman metric is given by

$$\omega_B\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^i}}\right) = \frac{|\frac{\partial}{\partial z^i} g_{i,x}(x)|^2}{|f_x(x)|^2},$$

where f_x is a function with L^2-norm $\|t\| = 1$ realizing the supremum of $|f(x)|$ among L^2-holomorphic functions $f \in H^0_{(2)}(M), \|f\| = 1$ on $\mathcal{T}$, and $g_{i,x}$ is a holomorphic function realizing supremum of $|\frac{\partial}{\partial z^i}(f)|^2$ among all $f \in H^0_{(2)}(M), \|f\| = 1, f(x) = 0$.

From assumption $B^n_a(x) \subset M \subset B^n_b(x)$, where $B^n_r(x)$ denotes a complex ball of radius r centered at x identified with 0 in $\mathbb{C}^n$. Let vol_o denote the Euclidean volume on $\mathbb{C}^n$. Clearly from the Mean Value Inequality

$$(f_x(x))^2 \leqslant \frac{\int_{B^n_a(x)} |f_x|^2}{\mathrm{vol}_o(B^n_r(x))} \leqslant \frac{\int_M |f_x|^2}{\mathrm{vol}_o(B^n_a(x))} = \frac{1}{a^{2n}\mathrm{vol}_o(B^n_1)}.$$

The constant function $h_1(x) = 1$ satisfies $h_1(1) = 1$ and

$$\|h_1\|^2 = \mathrm{vol}_o(M) \leqslant \mathrm{vol}_o(B^n_b) = b^{2n}\mathrm{vol}_o(B^n_1).$$

Hence $|f_x(x)| \geq \frac{1}{[(b)^{2n}\mathrm{vol}_o(B_1^n)]^{\frac12}}$. We conclude that

$$[\frac{1}{a^{2n}\mathrm{vol}_o(B_1^n)}]^{\frac12} \geq |f_x(x)| \geq [\frac{1}{b^{2n}\mathrm{vol}_o(B_1^n)}]^{\frac12}.$$

Let V_i be the complex line generated by $\frac{\partial}{\partial z^i}$ in $\mathbb{C}^n$. Then from Generalized Cauchy Inequality and Mean Value Inequality,

$$\begin{aligned}
|\frac{\partial}{\partial z^i}g_{i,x}(x)| &\leqslant \frac{\int_{\partial(B^n_{\frac a2}(x))\cap V_i}|g_{i,x}(y)|dy}{(\frac a2)^2} \leqslant \frac{[\int_{\partial(B^n_{\frac a2}(x))\cap V_i}|g_{i,x}(y)|^2dy]^{\frac12}[2\pi\frac a2]^{\frac12}}{(\frac a2)^2}\\
&\leqslant \frac{[\int_{\partial(B^n_{\frac a2}(x))\cap V_i}dy\int_{B^n_{\frac a2}(y)}|g_{i,x}(w)|^2d\mathrm{vol}_o(w)]^{\frac a2}[\frac\pi2]^{\frac12}}{(\frac a2)^2[\mathrm{vol}_o(B^n_{\frac a2})]^{\frac12}}\\
&\leqslant \frac{[\int_{\partial(B^n_{\frac a2}(x))\cap V_i}dy\int_{M}|g_{i,x}(w)|^2d\mathrm{vol}_o(w)]^{\frac12}[\frac\pi2]^{\frac12}}{(\frac a2)^2[\mathrm{vol}_o(B^n_{\frac a2})]^{\frac12}} \leqslant \frac{\frac\pi2}{(\frac a2)^{2+n}[\mathrm{vol}_o(B_1^n)]^{\frac12}}.
\end{aligned}$$

On the other hand the function $h_{i,x}=z_i$ satisfies $\frac{\partial}{\partial z^i}h_{i,x}=1$ and $h_{i,x}(0)=0$. As $\int_{B_1^n}|z_i|^2=\frac{1}{n+1}\mathrm{vol}(B_1^n)$, we know that

$$\|h_{i,x}\|^2 \leqslant \int_{B_b^n}|z_i|^2 = b^{2n+1}\int_{B_1^n}|z_i|^2 \leqslant \frac{1}{n+1}b^{2n+1}\mathrm{vol}_o(B_1^n).$$

Hence the function $k_{i,x}:=\frac{h_{i,x}}{\|h_{i,x}\|}$ satisfies $|\frac{\partial}{\partial z^i}k_{i,x}|=\frac{\sqrt{n+1}}{[b^{2n+1}\mathrm{vol}_o(B_1^n)]^{\frac12}}$, $k_{i,x}(0)=0$ and $\|k_{i,x}\|^2=1$.

We conclude as before that

$$\frac{\frac\pi2}{(\frac a2)^{n+2}[\mathrm{vol}_o(B_1^n)]^{\frac12}} \geq |\frac{\partial}{\partial z^i}g_{i,x}(x)| \geq \frac{\sqrt{n+1}}{b^{n+\frac12}[\mathrm{vol}_o(B_1^n)]^{\frac12}}.$$

Combining the above estimates for $f_x(x)$ and $g_{i,x}(x)$, we arrive at

$$\frac{2\pi}{a^2}(\frac{2b}{a})^n \geq \sqrt{g_B(x,\frac{\partial}{\partial z^i})} \geq \sqrt{\frac{n+1}{b}}(\frac ab)^n.$$

Since $a \leqslant \sqrt{g_K(x,\frac{\partial}{\partial z^i})} \leqslant b$ from Ahlfors Schwarz Lemma, we conclude that

$$\frac{2\pi}{a^3}(\frac{2b}{a})^n\sqrt{g_K(x,\frac{\partial}{\partial z^i})} \geq \sqrt{g_B(x,\frac{\partial}{\partial z^i})} \geq \sqrt{\frac{n+1}{b^2}}(\frac ab)^n\sqrt{g_K(x,\frac{\partial}{\partial z^i})}.$$

As $a^2 \leqslant g_K(x,V)$, g_K is non-degenerate on M. The earlier argument estimating g_B by g_K from below then implies that g_B is non-degenerate. Hence g_B is a Kähler metric.

We prove now that g_B is complete. If g_B is incomplete, it follows that there is a geodesic γ of finite length l from a fixed point $x_o \in M$ approaching to a point y on ∂M. In particular, given any preassigned number $\epsilon > 0$, we can choose a point z on γ so that the distance $d_B(z,y)=\lim_{w\to y}d_B(z,w)\leqslant\epsilon$. On the other hand, the above discussions relating g_B to g_K actually shows that the distance $d_B(z,\partial B_a(x))$ with respect to the Bergman metric is at least $a\cdot k_1$. This clearly leads to a contradiction by choosing $\epsilon < a\cdot k_1$. Hence g_B is complete.

Lemma 2. *M is a pseudoconvex domain.*

Proof Fix a realization of M as a bounded domain Ω in $\mathbb{C}^n$. From the previous lemma, the Bergman metric $\sqrt{-1}\partial\overline{\partial}\log K_{B,\Omega}$ is positive definite, here $K_{B,\Omega} = \sum_i |f_i(z)|^2$ is the potential of the Bergman metric on Ω expressed in terms of a unitary basis $\{f_i\}$ of the space of L^2-holomorphic functions on Ω. Clearly, $K_{B,\Omega}$ is a strictly plurisubharmonic function on M. We need only to prove that $K_{B,\Omega}$ blows up along any sequence of points approaching the boundary of Ω.

For a point $x \in \Omega \cong M$, let us still use the notation φ_x for the uniformizing coordinate charts for x as defined in the Introduction. Let $K_{B,\varphi_x(\Omega)}$ be the potential of the Bergman metric on $\varphi_x(\Omega)$. Clearly in terms of the Jacobian of the transition functions,

$$K_{B,\Omega} = K_{B,\varphi_x(\Omega)}|J(\varphi_x)|^2.$$

From the proof of Lemma 1, we know that

$$\begin{aligned} K_{B,\varphi_x(\Omega)}(y,y) &= \sup_{f\in H^0_{(2)}(\varphi_x(\Omega)),\|f\|_{\varphi_x(\Omega))}=1} |f(y)|^2 \\ &\geq \frac{1}{\int_{B_b(0)} 1} \end{aligned}$$

is bounded from below. Hence it suffices for us to prove that $|J(\varphi_x)|$ blows up for x approaching $\partial\Omega$.

Recall from definition that $\varphi_x^{-1} : \varphi_x(\Omega) \to \Omega$ is a biholomorphism and $\varphi_x^{-1}(0) = x$. We claim that as $x \to \partial\Omega$, the smallest eigenvalue μ_x of the Jacobian matrix $J(\varphi_x^{-1})|_0$ at 0 approaches to 0.

To prove the claim, we assume for the sake of proof by contradiction that there exists a sequence of points $x_i \in \Omega$ with Euclidean distance $d(x_i, \partial\Omega) = \epsilon_i \to 0$ but $\mu_{x_i} \geqslant c_1$ for some constant $c_1 > 0$. First of all, we observe by applying the generalized Cauchy estimates to φ_x^{-1} on $B_a(0)$ that every derivative of φ_x^{-1} is bounded from above by some constant independent of x. In particular, all second derivatives of φ_x^{-1} with respect to the coordinate vectors on $B_a(0)$ are bounded from above by a constant $c_2 > 0$. Let now ℓ_i be a line segment in Ω realizing the Euclidean distance between x_i and $\partial\Omega$, so that for $y_i \in \ell_i \cap \partial\Omega$, $d(x_i, y_i) = \epsilon_i$. The complexification of ℓ_i is a complex line $\ell_{i,\mathbb{C}}$ intersecting Ω. After a linear change of coordinate, we may assume that $\ell_{i,\mathbb{C}}$ is defined by $\zeta_2 = \cdots = \zeta_n = 0$. We may also assume that $\zeta_1 = 0$ at x_i. Writing $\zeta = t + iu$ in terms of real and imaginary part, we may assume without loss of generality that ℓ_i lies on the real axis defined by $u = 0$ and parametrized by t for $0 \leqslant t \leqslant \epsilon_i$. Hence the end point on $\partial\Omega$ is given by $\zeta_i = 0$, $i \geqslant 2$, and $\zeta_1(y_i) = \epsilon_i$.

As φ_{x_i} is a biholomorphism, the image $\widetilde{\ell_i} := \varphi_{x_i}(\ell_i \cap \Omega)$ is a real curve on $\varphi_{x_i}(\Omega)$ with 0 as an endpoint. Assume that $\widetilde{\ell_i}$ is parametrized by a unit speed parameter s so that $\widetilde{\ell_i}(0) = 0$ on $\varphi_{x_i}(\Omega)$. Since $\varphi_{x_i}^{-1}(B_a(0))$ intersects ℓ_i on Ω, we know that the length of $\widetilde{\ell_i}$ in φ_{x_i} is greater than a. Let $r = \min(a, \frac{c_1}{4c_2})$. Denote by $\lambda(s)$ the minimal eigenvalue of $J(\varphi_x^{-1})$ at $\widetilde{\ell}(s)$. $\lambda(0) = \mu_{x_i}$ from definition. From the Mean Value Theorem in calculus, it is clear that for $0 \leqslant s \leqslant r$, the minimal eigenvalue at s satisfies

$$\lambda(s) \geqslant \lambda(0) - c_2 s \geqslant c_1 - c_2 r \geqslant \frac{c_1}{2}.$$

It follows that $\frac{dt}{ds} \geqslant \frac{c_1}{2}$ for $0 \leqslant s \leqslant r$. Hence the length of ℓ_i is at least $\frac{c_1}{2}r$, a constant independent of x_i. Clearly this contradicts the assumption that the length of ℓ_i, which is $d(x_i, \partial\Omega)$, is ϵ_i and $\epsilon_i \to 0$ as $i \to \infty$. The claim is proved.

As mentioned above, each eigenvalue of the Jacobian of φ_x^{-1} is bounded from above by a constant c_3 for all points $x \in \Omega$. Moreover, the smallest of them approaches to 0 as $x \to \partial\Omega$. Since the determinant $|J(\varphi_x^{-1})|$ is just the product of all the eigenvalues of the $J(\varphi_x^{-1})$, we conclude that $|J(\varphi_x^{-1}| \to 0$ as $x \to \partial\Omega$. Hence $|J(\varphi_x)|$ tends to ∞ as $x \to \partial\Omega$.

Remark It was pointed out by the referee that the argument essentially showed that the trace of the Bergman kernel $K(x,x)$ of a uniform squeezing domain was bounded from below by c/d, where $d = d(x, \partial\Omega)$ is the Euclidean distance to the boundary of Ω and c is a constant. Later on we will see that as a consequence of Theorem 1 and 2, the estimates can be improved to $c/(d^2(-\log d)^2)$ as stated in Corollary 3.

We may now complete the proof of Theorem 1.

Proof of Theorem 1 (a) and (b) follow from Lemma 1 and Lemma 2. (c) follows from the work of Cheng-Yau and Mok-Yau on Kähler-Einstein metrics (cf. [MY]).

§3 Metric properties

We say that we metrics g_1 and g_2 are equivalent or quasi-isometric on a domain Δ, denoted by $g_1 \sim g_2$, if there exists a constant $c > 0$ such that $\frac{1}{c} g_2 \leqslant g_1 \leqslant c g_2$.

Proposition 3. *The invariant metrics on a uniformly squeezing domain satisfy* $g_C \sim g_K \sim g_B \sim g_{KE}$. *More precisely,*

$$\begin{aligned}
\frac{a}{b} g_K \leqslant \quad & g_C \quad \leqslant g_K, \\
\frac{a}{b} g_K \leqslant \quad & g_B \quad \leqslant [\frac{2\pi}{a^3}(\frac{2b}{a})^n]^2 g_K, \\
\frac{a^2}{b^2 n} g_K \leqslant \quad & g_{KE} \quad \leqslant \frac{b^{4n-2} n^{n-1}}{a^{2n-2}} g_K.
\end{aligned}$$

Proof Since the proof is very similar to the proof of Theorem in [Y3], we would just give a brief outline.

It follows from Ahlfors Schwarz Lemma that $g_C \leqslant g_K$. On the other hand, from definition of g_K and g_C and the inclusions $B_a^n(x) \subset \varphi(M) \subset B_b^n(x)$, we conclude for any tangent vector $v \in T_v M$ that $\sqrt{g_K(x,v)} \leqslant \frac{1}{a}$ and $\sqrt{g_C(x,v)} \geq b$. Hence $g_B \leqslant \frac{b}{a} g_C$. Hence $g_B \geqslant \frac{a}{b}$ from the above discussions.

The upper bound of g_B by g_K is already given in the proof of Lemma 1. On the other hand, as observed by Look and Hahn (cf. [H]), it follows by expressing g_B in terms of extremal functions that $g_K \geq g_C$.

To compare g_{KE} and g_K, we normalized the Poincaré metric on $B_r(0)$ so that the potential is $\log(r^2 - |z|^2)$. The resulting metric is a Kähler-Einstein metric of Ricci curvature $-\frac{2n+2}{r^2}$ with constant holomorphic sectional curvature $-4 < 0$. Then $g_{KE}^{B_r}(v, \overline{v}) = \frac{1}{r^2} = g_K^{B_r}(x,v)$. It follows from definition that on B_r, $g_K^{B_r}(0,v) = g_C^{B_r}(0,v) = [\frac{1}{r}]^2 = g_{KE}^{B_r}(v,\overline{v})$. $B_a^n \subset M_S \subset B_b^n$. Let us denote the volume form of g by $\mu(g)$. Applying Schwarz Lemma of Mok-Yau [MY] to the first inclusion with respect to the Kähler-Einstein metrics $g_{KE}^{B_a^n}$ and g_{KE}^M on B_a^n and M of Ricci

curvature $-\frac{2n+2}{(a)^2}$ and $-(2n+2)$ respectively, we get

$$\begin{aligned}\mu(g_{KE}^M) &\leqslant a^{2n}\mu(g_{KE}^{B_a^n}) = a^{2n}\mu(g_K^{B_a^n}) \\ &= b^{2n}\mu(g_K^{B_b^n}) \leqslant b^{2n}\mu(g_K^M)\end{aligned}$$

Applying Schwarz Lemma of [R] to g_{KE}^M which has constant Ricci curvature $-(2n+2)$ and $g_{KE}^{B_b^n}$ which has constant holomorphic sectional curvature -4, we conclude that

$$g_{KE}^M \geq \frac{1}{n} g_{KE}^{B_b^n} = \frac{1}{n} g_K^{B_b^n} \geq \frac{a^2}{nb^2} g_K^{B_a^n} \geq \frac{a^2}{nb^2} g_K^M.$$

Let $\nu_i > 0, i = 1, \ldots, n$ be the eigenvalues of g_{KE}^M with respect to g_K^M. We conclude from the second estimate that $\nu_i \geq \frac{a^2}{nb^2}$ for all i, and from the first statement that $\prod_{i=1}^n \nu_i \leqslant b^{2n}$. It follows that $\frac{b^{4n-2}n^{n-1}}{a^{2n-2}} \geq \nu_i \geq \frac{a^2}{b^2 n}$. Hence

$$(\frac{b}{a})^{2n-2} n^{n-1} g_K \geq g_{KE} \geq \frac{a^2}{b^2 n} g_K.$$

This concludes the proof of the proposition.

Proposition 4. *(a). There exists a constant $c_N^{g_{KE}}$ depending only on the order of differentiation N such that $\|\nabla_{i_1,\cdots i_N}^{g_{KE}} R_{g_{KE}}\| \leqslant c_N^{g_{KE}}$ for any covariant derivatives $\nabla_{i_1,\cdots i_N}^{g_{KE}}$. Consequently, the curvature tensor of g_{KE} and any order of covariant derivatives of the curvature tensor is bounded by a uniform constant. Furthermore, the injectivity radius of g_{KE} is bounded uniformly from below on M.*
(b). The same conclusion is true for the Bergman metric g_B.

Proof (a). We denote g_{KE} simply by g in this part of proof. For each fixed point $x \in M$, there exists a uniformizing squeezing coordinate system given by $B_a(0) \subset \varphi(M) \subset B_b(0) \subset \mathbb{C}^n$, where $\varphi(x) = 0$. We would derive our estimates on such coordinate neighborhoods. By a unitary change of coordinates, we may assume that $g_{i\bar{j}}(x)$ is diagonal at $x = 0$. Furthermore, from Lemma 3, we know that on $B_{\frac{a}{2}}(0)$, $g_{KE} \sim g_K \sim g_o$. Hence in measuring the magnitude of a derivative with respect to $g = g_{KE}$, it is up to some uniform constant the same as measuring with respect to the Euclidean metric g_o. We need the following technical estimates.

Lemma 3. *Let $g = g_{KE}$ be the Kähler-Einstein metric on a domain M with uniform squeezing properties. Then all the covariant derivatives of the coordinate vector fields in terms of the uniform squeezing coordinate systems are uniformly bounded on M by a constant depending on the order of differentiations.*

Proof In terms of the uniform squeezing coordinate system, Lemma 3 is equivalent to the boundedness of any order of derivatives of the metric coefficient $g_{i\bar{j}}$ with respect to the coordinate vectors.

In the following, we denote by $c_i, c_{k,m}$ and $c'_{k,m}$ constants which are independent of $x \in M$. The Kähler metric satisfies Einstein equation

$$\partial_i \overline{\partial}_j \log|\det(g)| = c g_{i\bar{j}} \tag{0.1}$$

on $B_a(x)$ for $x \in M$.

Note that $g_{i\bar{j}}$ coming from solution of Monge-Ampère equation is smooth from the standard results in Kähler-Einstein equations (cf. [Au], chapter 7). In fact, it

would also follow from Proposition 2 together with Theorem 3.56 of [Au]. Taking trace with respect to the Euclidean metric, we get

$$\Delta_o \log|\det(g)| = c g_o^{i\bar{j}} g_{i\bar{j}} \tag{0.2}$$

on $B_a(x)$.

Recall also from Proposition 3 that on $B_{a/2}(x)$, $g_{i\bar{j}}$ is quasi-isometric to the Euclidean metric $|(g_o)_{i\bar{j}}| = \delta_{ij}$, where δ_{ij} are Kronecker's delta.

In terms of the usual notion used in [GT], let us denote by $H_k = W_{k,2}$ and $C_{k,\alpha}$ the spaces of functions on B_a which are bounded with respect to the Sobolev norm and Hölder norm on $B_a(x)$ respectively. Applying Calderon-Zymund's estimates in Proposition 2 to the Einstein equation, we get

$$\|\log|\det(g)|\|_{H_2} \leqslant c_1[\|\log|\det(g)|\|_{H_0} + \|g_o^{i\bar{j}} g_{i\bar{j}}\|_{H_0}] \leqslant c_2,$$

here again we used Proposition 3.

Observe that on each point $y \in B_a(x)$, there exists a unitary matrix A_y such that $A_y g \overline{A}_y^t$ is a diagonal matrix. As $|\det A_y| = 1$ from definition, we may assume that g is diagonal at y for our computation involving $\det(g)$. Hence $|\det(g)| = \prod_{i=1}^n g_{i\bar{i}}$ at y. The Einstein equation gives rise to $\partial_i \overline{\partial}_i \log|\det(g)| = c g_{i\bar{i}}$. Let X be any vector field of unit length coming from linear combination of coordinate vector fields. Let D_X denote derivative in the direction of X. Then by applying D_X to the above equation

$$\partial_i \overline{\partial}_i D_X \log|\det(g)| = c D_X g_{i\bar{i}} = c g_{i\bar{i}} D_X \log|(g_{i\bar{i}})|.$$

Taking the trace by g and summing over all $i = 1, \ldots, n$, we get

$$\sum_{i=1}^n g^{i\bar{i}}(y) \partial_i \overline{\partial}_i D_X \log|\det(g)| = c \sum_{i=1}^n D_X \log|(g_{i\bar{i}})(y)| = D_X \log|\det(g)|(y).$$

We obtain $\Delta_g D_X \log|\det(g)| = D_X \log|\det(g)|(y)$. As $g_{i\bar{j}}$ is uniformly quasi-isometric to $(g_o)_{i\bar{j}}$, the same Schauder estimate allow us to conclude that

$$\|D_X \log|\det(g)|\|_{H_2} \leqslant c_3[\|D_X \log|\det(g)|\|_{H_0}] \leqslant c_4$$

after applying the earlier bound on $\log|\det(g)|\|_{H_2}$. As X is arbitrary, this implies

$$\|\log|\det(g)|\|_{H_3} \leqslant c_4.$$

Clearly the bootstrapping argument implies that for each positive integer m, there exists a constant c_m independent of x such that

$$\|\log|\det(g)|\|_{H_m} \leqslant c_m.$$

Applying the Sobolev Estimates to $B_{a/2}(x)$, we conclude that

$$\|\log|\det(g)|\|_{C_{k,m}} \leqslant c_{k,m}$$

for some constant $c_{k,m}$ independent of x. Applying to equation (0.2), we conclude that

$$\|g_{i\bar{j}}\|_{C_{k,m}} \leqslant c'_{k,m}$$

for some constant $c'_{k,m}$ independent of x. This concludes the proof of the Lemma.

We may now complete the proof of (a). We compute in terms of the coordinate vectors in the uniform squeezing coordinate system. A covariant derivative of the curvature tensor is a Euclidean derivative modified by an addition term coming from Christopher symbol, which are linear combinations of the derivatives of the

metric tensor with respect to coordinate vectors. It follows easily from Lemma 3 and by induction that the norm of an n-th order derivative of the metric tensor with respect to the coordinate vectors is bounded by a constant depending on n but independent of x. Hence the first statement of (a) follows.

Since the curvature tensor of g_{KE} appears as sum of some second order and first order derivatives of the metric tensor with respect to coordinate vectors, it is clear that the curvature tensor is bounded uniformly. Similarly, any order N covariant derivatives of the curvature are linear combination of expressions involving up to $N+2$ order of derivatives of the metric tensor with respect to the coordinate vectors. We conclude that any such derivative is bounded by a constant depending on N.

From the boundedness in curvature, we conclude immediately that the conjugate radius is bounded from below by an absolute constant. Furthermore we note that there exists a $\epsilon > 0$ such that a geodesic loop l of length less than ϵ based at x does not exist. Suppose on the contrary such a geodesic exists and the incoming and outcoming geodesic segment span an angle θ, which has to be positive, at x. Then we may find two points P_1, P_2 on l near cut-locus of x such that the distance of their preimages on the tangent space with induced metric by the exponential map at x is bounded from below by $\theta\epsilon$, but clearly not on M. This is clearly a contradiction for ϵ sufficiently small and the fact that the metric is quasi-isometric to the Euclidean one.

(b). To consider the derivatives of the Bergman metric, again we consider the uniformizing squeezing coordinates and let $K_x(z,w) = \sum_i f_i(z)\overline{f_i(w)}$ be the coefficient of the Bergman kernel on $\varphi_x(M)$ which is holomorphic in z and conjugate holomorphic in w. Let $\overline{M}$ be the set M equipped with the conjugate complex structure. Writing $w_i = \overline{u}_i$, we conclude that $K_x(z,\bar{u})$ is holomorphic on $M \times \overline{M}$ with respect to the complex structures on M and $\overline{M}$ respectively. The restriction $K(z,w)$ to $w = z$ is precisely the potential for the Bergman metric.

Let $A = B_{\frac{a}{2}} \times B_{\frac{a}{2}}$. Let D be a differential operator involving compositions of the coordinate derivatives. By Generalized Cauchy Inequality, it follows easily that all the higher derivatives $|[D\frac{\partial}{\partial z^i}\frac{\partial}{\partial w^j}\log K](z_o,\overline{u}_o)|$ of the metric at the origin are controlled up to a constant depending on D by $|K(z,\overline{u})|$ for (z,u) lying on the boundary $\partial(A)$. Clearly $|K_x(z,\overline{u})|^2 \leqslant K_x(z,z)K_x(\overline{u},\overline{u})$ by the Cauchy-Schwarz Inequality for $(z,u) \in \partial A$.

In terms of the peak function f_z at $z \in \varphi(M)$ mentioned before, we obtain

$$\begin{aligned} K_x(z,z) &= |f_z|^2 \leqslant \frac{1}{\mathrm{vol}(B_{\frac{a}{2}}(z))}\int_{\mathrm{vol}(B_{\frac{a}{2}}(z))}|f_z|^2 \\ &\leqslant \frac{1}{\mathrm{vol}(B_{\frac{a}{2}})}, \end{aligned}$$

since the L^2-norm of f_z is 1. The same bound is applicable to $K_x(\overline{u},\overline{u})$. Restricting to the twisted diagonal given by $u = \bar{z}$, it follows immediately that the curvature tensor and all their derivatives are bounded with respect to the Euclidean metric on $B_{\frac{a}{2}}(z) \subset M$. As g_B is uniformly quasi-isometric to g_o on $\mathrm{vol}(B_{\frac{a}{2}})$, we conclude that all the derivatives of the curvature are uniformly bounded for the Bergman metric. As in part (a), the finishes the proof of (b).

Proposition 5. *(a). M is Kähler-hyperbolic with respect to g_{KE}.*
(b). The same is true for g_B.

The proof of the proposition depends on the following lemma.

Lemma 4. *Let $g = g_{KE}$. Fix $x, y \in M$. Let W be a $(1,0)$-vector at $0 \in \varphi_y(M)$ Let $\varphi_{y,x} : \varphi_y(M) \to \varphi_x(M)$ be the biholomorphic mapping given by $\varphi_x \circ \varphi_y^{-1}$. Then $\frac{|\partial_W \log |J(\varphi_{y,x})|^2|}{\sqrt{g(W,\overline{W})}}(0) \leqslant C$ for some constant C independent of x and y.*

Proof Let $z^i, w^j, i, j = 1, \ldots, n$ be the local coordinates on $\varphi_y(M)$ and $\varphi_x(M)$ respectively. For simplicity, we also denote $\varphi_{y,x}$ by φ since x and y are fixed in this proof. Let us choose the coordinate at $\varphi_x(M)$ such that $g(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}})$ is diagonal at $\varphi(0)$.

It suffices for us to show that for each $k = 1, \ldots, n$, $\frac{|\frac{\partial}{\partial z^k} \log |J(\varphi_{y,x})|^2|}{\sqrt{g(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^k})}}(0) \leqslant C$. Clearly,

$$|J(\varphi)(z)|^2 = \frac{\det(g(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}))(z)}{\det(g(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}))(\varphi(z))}.$$

Hence

$$\frac{|\frac{\partial}{\partial z^k} \log |J(\varphi_{y,x})|^2|}{\sqrt{g(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^k})}}(0) \leqslant [\frac{1}{\sqrt{g(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^k})}} \frac{\partial}{\partial z^k} |\log|\det(g(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}))||]_{z=0}$$
$$+ [\frac{1}{\sqrt{g(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^k})}} \frac{\partial}{\partial z^k} |\log|\det(g(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}))(\varphi(z))||]_{z=0}.$$

For the first term, applying Schauder type estimates to the chart $\varphi_y(M)$ and using the fact that $g = g_{KE}$ is Kähler-Einstein, we conclude as in the proof of Proposition 3 that

$$[|\frac{1}{\sqrt{g(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^k})}} \frac{\partial}{\partial z^k} |\log|\det(g(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}))(z)||]_{z=0} < c_1,$$

Note that quasi-isometry of $g = g_{KE}$ and the Euclidean metric is used here.

For the second term, we rewrite $w = \varphi(z)$ and use Chain rule to rewrite

$$\begin{aligned}
&[\frac{1}{\sqrt{g(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^k})}} \frac{\partial}{\partial z^k} |\log|\det(g(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}))(\varphi(z))||]_{z=0} \\
= \quad & [\frac{1}{\sqrt{\sum_{i,j} g(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}) \frac{\partial w^i}{\partial z^k} \overline{\frac{\partial w^j}{\partial z^k}}}} |\sum_l \frac{\partial w^l}{\partial z^k} \frac{\partial}{\partial w^l} |\log|\det(g(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}))(\varphi(z))|]_{z=0} \\
= \quad & \frac{1}{\sqrt{\sum_i g(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^i}}) |\frac{\partial w^i}{\partial z^k}|^2(0)}} [|\sum_l \frac{\partial w^l}{\partial z^k}(0) \frac{\partial}{\partial w^l} |\log|\det(g(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}))(w)|]_{w=\varphi(0)}.
\end{aligned}$$

Applying now Schauder's estimate to $\varphi_x(M)$ and using the fact that g is Kähler-Einstein and g is quasi-isometric to the Euclidean metric on $\varphi_x(M)$ again, we

conclude that $|\frac{\partial}{\partial w^l}|\log|\det(g(\frac{\partial}{\partial w^i},\overline{\frac{\partial}{\partial w^j}}))(w)|_{w=\varphi(0)} \leqslant c_2$. Hence

$$\frac{1}{\sqrt{\sum_i g(\frac{\partial}{\partial w^i},\overline{\frac{\partial}{\partial w^i}})(\varphi(0))|\frac{\partial w^i}{\partial z^k}|^2(0)}}|\frac{\partial w^l}{\partial z^k}(0)\frac{\partial}{\partial w^l}|\log|\det(g(\frac{\partial}{\partial w^i},\overline{\frac{\partial}{\partial w^j}}))(\varphi(0))| \leqslant c_3$$

and we conclude that

$$\frac{|\frac{\partial}{\partial z^k}\log|J(\varphi_{y,x})|^2|}{\sqrt{g(\frac{\partial}{\partial z^k},\frac{\partial}{\partial \bar{z}^k})}}(0) \leqslant c_1 + c_3.$$

As k is arbitrary, this concludes the proof of the Lemma.

Remark Lemma 4 gives a proof of the part of Proposition 5 of [Y4] about boundedness of the first derivative of the Jacobian, showing that it corresponds to general properties of uniform squeezing domains. The last sentence in Proposition 5 of [Y4] that $|J(\Phi_{y,x})| = 1$ is incorrect and was not used in the subsequent arguments in [Y4].

Proof of Proposition 5 (a) Fix a point $x \in M$ and consider the uniform squeezing coordinates φ_x associated to it, so that $B_a(0) \subset \varphi_x(M) \subset B_b(0)$. The Kähler form ω_{KE} associated to the Kähler-Einstein metric $g_{KE,x}$ satisfies

$$\sqrt{-1}\partial\overline{\partial}\log\det g_{KE} = c\omega_{KE}$$

for some negative constant c. Left hand side of the above expression is independent of the particular coordinate φ_x that we are using. Note that the determinant $\det(g_{KE}(\frac{\partial}{\partial z^i},\overline{\frac{\partial}{\partial z^j}}))$ depends on the local coordinates $\varphi_x(M)$. We denote the quantity by $\det(g_{KE,x})$. In this way we may regard $h_x = \sqrt{-1}\partial\log\det g_{KE,x}$ as the potential one form to satisfy $dh_x = \omega$, here note that h_x depends on our fixed base point x. Again, on $B_{\frac{a}{2}}(0)$, the equation is

$$\Delta_{g_{KE,x}}\log\det g_{KE,x} = c.$$

Since $g_{KE,x} \sim g_K \sim g_o$ on $B_{\frac{a}{2}}(0)$, the above equation is a strongly elliptic equation with uniformly bounded coefficients. It follows from Proposition 3 that we have a bound $|d\log\det g_{KE,x}| \leqslant C$ for some uniform constant C. We conclude that $|\sqrt{-1}\partial\log\det g_{KE,x}| < C$ and hence

$$|h_x|_{g_{KE,x}} = |\sqrt{-1}\partial\log\det g_{KE,x}(y)|_{g_{KE,x}} < C_1$$

for some uniform constant C_1 and all $y \in B_{\frac{a}{2}}(0)$.

Now we need to worry about points $y \in \varphi_x(M) - B_{\frac{a}{2}}(0)$. For such cases, we consider the uniform squeezing coordinate φ_y as well, here we identify y with $\varphi_x^{-1}(y)$ to simplify our notations. $\varphi_{y,x} = \varphi_y \circ \varphi_x^{-1}$ is the biholomorphism from $\varphi_x(M)$ to $\varphi_y(M)$. We have correspondingly $\det(g_{KE,x}(z)) = \det(g_{KE,y}(w))|J(\varphi_{y,x})|^2$. Let $Y \in T_0(\varphi_y(M))$ and $X = (\varphi_{y,x})_*Y \in T_{\varphi_{y,x}(0)}(\varphi_x(M)$. Then

$$\begin{aligned} h_x(X) &= \sqrt{-1}\partial_X\log\det(g_{KE,x}) = \sqrt{-1}\partial_Y\log\det(g_{KE,y}) + \sqrt{-1}\partial_X\log|J(\varphi_{y,x})|^2 \\ &= h_y + \sqrt{-1}\partial_X\log|J(\varphi_{y,x})|^2. \end{aligned}$$

Clearly it follows from the earlier argument that $|h_y|_{g_{KE,y}} < c_2$ for some uniform constant c_2. Lemma 4 also shows that $|\partial_X\log|J(\varphi_{y,x})||^2_{g_{KE,y}} < c_3$. Proposition 3 for g_{KE} now follows from combining these two estimates.

(b) Fix our point $x \in M$ as before. The Kähler form of g_B can be written as $\omega_B = d\eta_x$, where $\eta_x = -\sqrt{-1}K_x^{-1}\partial K_x$ and $K_x \sum_j |f_j(z)|^2$ by taking over unitary basis of holomorphic functions on the Hilbert space of L^2-holomorphic functions on $\varphi_x(M)$. Proposition 3b above implies that $|\eta_x(y)|_{g_B} < c_4$ for some constant c_4 and every point $y \in B_{\frac{a}{2}}(0) \subset \varphi_y(M)$.

For a point $y \in \varphi_x(M) - B_{\frac{a}{2}}(0)$, we note that the potentials of g_B satisfies $K_x = K_y|J(\varphi_{y,x})|^2$. Hence

$$\eta_x(X) = \eta_y(Y) + \sqrt{-1}\partial_X \log |J(\varphi_{y,x})|^2$$

similar to the derivation in (a). From the previous paragraph, $|\eta_y|_{g_B} < c_5$ for some uniform constant $c_5 > 0$. From Proposition 3b again, $|\partial_X \log |J(\varphi_{y,x})(y)|^2_{g_B} \leqslant c_6|\partial_X \log |J(\varphi_{y,x})(y)|^2_{g_o}| \leqslant c_7$ for some constant $c_7 > 0$. (b) follows by combining the previous two estimates.

Lemma 5. *Let x be a fixed point on M. Expressed in terms of the uniform squeezing coordinate system $\varphi_x(M)$, $|\det(g_{KE})|^{-\alpha}$ for α sufficiently small is a bounded plurisubharmonic exhaustion function on M.*

Proof Denote by $|g_{KE}| = \det(g_{KE})$ the determinant of g_{KE} in local coordinates. Direct computation yields

$$\begin{aligned} & \sqrt{-1}\partial\overline{\partial}(-|g_{KE}|^{-\alpha}) \\ = \quad & -\alpha(\alpha+1)|g_{KE}|^{-\alpha-2}\sqrt{-1}\partial|g_{KE}| \wedge \overline{\partial}|g_{KE}| + \alpha|g_{KE}|^{-\alpha-1}\sqrt{-1}\partial\overline{\partial}|g_{KE}| \\ = \quad & \alpha|g_{KE}|^{-\alpha}\sqrt{-1}[(\alpha+1)\partial\overline{\partial}(\log|g_{KE}|) - \partial\log|g_{KE}| \wedge \overline{\partial}\log|g_{KE}|]. \end{aligned}$$

Applying Proposition 5 and noting that $\log|g_{KE}|$ is up to a constant the potential of g_{KE} on any realization of M as a bounded domain, $|\sqrt{-1}\partial\log|g_{KE}|\wedge\overline{\partial}\log|g_{KE})|| \leqslant c\sqrt{-1}\partial\overline{\partial}\log(|g_{KE}|)$ for some constant $c > 0$. It suffices for us to choose $\alpha > \frac{1}{c} - 1$ to conclude the proof of the lemma.

Proof of Theorem 2 (a) follows from Proposition 3. (b), (c) and (d) follows from Proposition 4. (e) follows from Proposition 5. (f) follows from Lemma 5.

Remark We remark that finiteness in volume of g_{KE} is in fact equivalent to the quasiprojectiveness of our M. One direction is proved in the above corollary. For the other direction , as assume that $M_1 = M/\Gamma$ is a quasi-projective manifold. We may assume that $M_1 = \overline{M}_1 - D$ for some normal crossing divisor D after resolution of singularities if necessary. Hence neighborhoods of D in M_1 are covered by union of open sets U_i of the form $\Delta_1^a \times (\Delta_1^*)^b$, where Δ_1 is a Poincaré disk of radius 1 and Δ_1^* is a punctured disk of radius 1.

Equip each such $U_i = \Delta_1^a \times (\Delta_1^*)^b$, we consider a smaller open set $U_r = \Delta_r^a \times (\Delta_r^*)^b$ for $0 < r < 1$ with the restriction of the Poincaré metric g_P on U_i and apply the Schwarz Lemma of Mok-Yau [MY] to the embeddings of the inclusion of (U_i, g_P) into (M_1, g_{KE}), we conclude easily that the volume of (M_1, g_{KE}) is finite since the volume of each (U_i, g_P) is finite.

§4 Examples

We are going to show that examples in Proposition 1 do satisfy the uniform squeezing property.

Proof of Proposition 1 We will prove (a), (b), (d) and (e) first and leave the proof of (c) to the end.

(a) Bounded homogeneous domain M in $\mathbb{C}^n$.

Choose any point on M and translate the origin of $\mathbb{C}^n$ to that point. As it is bounded, it is contained in a ball $B_b(x)$. Since 0 lies in the interior of M, there exists a ball of positive radius a such that $B_a(0) \subset M$. Hence $0 \in B_a(0) \subset M \subset B_b(x) \subset \mathbb{C}^n$. Let x be an arbitrary point on M. As M is homogeneous, there exists a biholomorphism of M moving x to 0, here we realize M as a fixed domain in $\mathbb{C}^n$. Hence balls of the same radii provide uniform squeezing coordinates for all $x \in M$.

(c) Bounded domains which cover a compact Kähler manifold

Assume that $M \subset B_c(x_o) \subset \mathbb{C}^n$ is a bounded domain covering a compact Kähler manifold N, where x_o is a fixed point on M. Let A be a fundamental domain of N in M. For each point $x \in A$, there exists a ball of radius r_x such that $B_{r_x}(x) \subset M$. Since A is relatively compact, $r = \inf_{x\in A} r_x > 0$. It is then clear that $B_a(x) \subset M \subset B_b(x)$ for each $x \in A$. Since each point $y \in M$ can be mapped biholomorphically to some point in A by the deck transformation group, it is clear that we get a (a, b) uniform squeezing coordinate.

(d) Teichmüller spaces $\mathcal{T}_{g,n}$ of compact Riemann surfaces of genus g with n punctures

This is a consequence of Bers Embedding Theorem described as follows (cf. [Ga]). Let S be a Riemann surface of genus g with n punctures representing a point $x \in \mathcal{T}_{g,n}$. Denote by $\mathcal{T}_S$ the Teichmüller space based at x. There exists an embedding $\Phi : \mathcal{T}_S \to \mathbb{C}^N$, so that $B^N_{\frac{1}{2}} \subset \mathcal{T}_S \subset B^N_{\frac{3}{2}}$, where $\mathbb{C}^N$ is identified with the space of holomorphic quadratic differentials based at S equipped with L^∞ norm, and $\Phi(x) = 0$, where $N = 3g - 3 + n$.

Hence the charts associated to Bers embedding provide us the uniform squeezing coordinates.

(b) Bounded smooth strongly convex domains

We give a step by step construction of the uniform squeezing coordinate systems.

(i) We observe the following fact. Suppose $C_1 = \partial B^1_a(x)$ and $C_2 = \partial B^1_b(y)$ are two circles in $\mathbb{C}$ of radii a and b meeting tangentially at one point. Assume that $B^1_a(x) \subset B^1_b(y)$. Let $w \in B^1_a(x)$ lying on the real line joining x and y. Then there exists a Möbius f mapping C_1 to itself, so that f is holomorphic on $B^1_b(y)$, $f(w) = 0$ and $f(C_2) \subset B^1_{2b}(0)$.

To see this, we may assume that $x = 0$ and $a = 0$ by rescaling. By a linear change of coordinates, we may also assume that $y = -b + 1$ lies on the real axis of $\mathbb{C}$. The fact follows by inspecting the explicit Möbius transformation $z \to (z - w)/(1 - z\overline{w})$.

(ii) We claim the following fact. Suppose $B_a(x) \subset B_b(y)$ are two balls in $\mathbb{C}^n$ and $\partial B_a(x)$ is tangential to $\partial B_b(y)$ at a point q. Let $w \in B_a(x)$ lying on the real line joining x and y. Then there exists a Möbius transformation ψ of $B_a(x)$, so that ψ is biholomorphic on $B_a(x)$, ψ is holomorphic on $B_b(y)$, $\psi(w) = 0$ and $\psi(B_b(y)) \subset B_{2b}(0)$.

To see this, after a linear change of coordinates, we may assume that the real line joining x and y is defined by $z_2 = \cdots z_n = 0$ and $Im(z_1) = 0$. As in (i), we may assume that $x = 0$ by an affine change change of coordinate, and $a = 1$ after

rescaling. Consider now the Möbius transformation given by

$$\psi(z_1, \cdots, z_n) = (\frac{z_1 - w}{1 - z_1\overline{w}}, \frac{\sqrt{1-|w|^2}}{1 - z_1\overline{w}} z_2, \cdots, \frac{\sqrt{1-|w|^2}}{1 - z_1\overline{w}} z_n).$$

The same computation as in (i) establishes the claim.

(iii) We now proceed to construct the uniform squeezing coordinate system. We are considering a C^2-strongly convex domain M in $\mathbb{C}^n$. Let $p \in \partial M$. Let U_p be a neighborhood of p. For a point $q \in U'_p := \partial M_p \cap U_p$, let $N_{U'_p}(q)$ be the real line which is normal to ∂M at q with respect to the Euclidean metric. As ∂M is C^2-smooth and M is convex, there exist point $x_{p,q}, y_{p,q} \in N_{U'_p}(q)$, and positive numbers $a_{p,q}$ and $b_{p,q}$ such that both $\partial B_{a_{p,q}}(x_{p,q})$ and $\partial B_{b_{p,q}}(y_{p,q})$ are tangential to $\partial M'_p$ at q and $B_{a_{p,q}}(x_{p,q}) \subset M \subset B_{b_{p,q}}(y_{p,q})$.

Replacing U_p by a slightly smaller relatively compact subset of itself if necessary, we may assume that $a_p = \liminf_{q \in U'_p} a_{p,q} > 0$ and $b_p = 2 \limsup_{q \in U'_p} b_{p,q} < \infty$. Let $x'_{p,q}$ be the unique point on the normal line $N_{U'_p}(q) \cap M$ at a distance a_p from q. Let $V_p = \cup_{q \in U'_p} B_{a_p}(x'_{p,q})$. From the above construction and from the claim in (ii), an (a_p, b_p)-uniform squeezing coordinate charts exists for V_p.

The union $\cup_{p \in \partial M} V_p$ covers a neighborhood of ∂M. From compactness of ∂M, we can choose a finite number of points $p_1, \cdots, p_N$ on ∂M such that $\cap_{i=1}^N V_{p_i}$ covers a neighborhood of ∂M. Let V_o be a relatively compact open subset of M containing $M - \cap_{i=1}^N V_{p_i}$ so that $\{V_i\}_{i=0,\ldots,N}$ gives a holomorphic covering of M. It is clear that there exists $0 < a_0 < b_0$ such that for each point $z \in V_0$, there exists a holomorphic coordinate charts with $B_{a_0}(z) \subset M \subset B_{b_0}(z)$. Let $a = \min(a_0, a_i, 1 \leqslant i \leqslant N)$ and $b = \max(b_0, b_i, 1 \leqslant i \leqslant N)$. It follows from our construction that the balls of radii a and b involved form an (a, b)-uniform squeezing coordinate system for M. Hence strictly convex domain with C^2 boundary satisfies the uniform squeezing property.

This concludes the proof of Proposition 1.

§5 Geometric consequences

In this section, we give a proof of Corollary 1, Corollary 2 and Theorem 3 as applications of Theorem 1 and 2. A proof for Theorem 4 is also explained.

Proof of Corollary 1 (a) and (c) follows from the argument in [Gr]. (b) is already proved in [M], once we know that the manifold involved is Kähler-hyperbolic.

Proof of Corollary 2 (a). It is given that $N = M/\Gamma$ is a compact complex manifold. Since we have already proved that there exists a Kähler-Einstein of negative scaler curvature on M which descends to N, it follows immediately that the canonical line bundle is ample and hence the variety is of general type.

(b). From assumption, M/Γ has finite volume with respect to the Kähler-Einstein metric g_{KE}. From Theorem 2a, g_{KE} is complete Kähler, with constant negative Ricci curvature and bounded Riemannian sectional curvature. Hence we may apply the results of [Y1], which relies on the earlier results of Mok-Zhong [MZ], to conclude that M_1 is quasi-projective.

We note that $h^i_{(2)}(M_1, lK_{M_1}) = 0$ for $l \geq 2$ from L^2-estimates as in the corresponding proof of Kodaira's Vanishing Theorem. Hence $h^i_{(2)}(M_1, lK_{M_1}) = \chi(M, lK_M)$.

It follows from a generalized form of the Riemann-Roch estimates of Demailly [De] as proved in [NT] that

$$\chi(M_1, lK_{M_1}) = \frac{l^n}{n!}\int_M c_1(K_M)^n + o(l^N).$$

Here we note that $c_1(K_M)$ is positive definite from construction. L^2-holomorphic sections of lK_M extends as a holomorphic section in $H^0_{\overline{M}_1}(\overline{M}_1, l(K_{\overline{M}_1}+D))$. This follows from the fact that they extend as L^2-sections and hence cannot have poles of order greater than 1 along any component of the compactifying divisor. We conclude that $\dim(H^0_{\overline{M}_1}(\overline{M}_1, l(K_{\overline{M}_1}+D))) \geq cl^N$ and hence that $(\overline{M}_1, D)$ is of log-general type. This concludes the proof of Corollary 2.

Proof of Theorem 3 It follows from Proposition 1 that a bounded domain which is the universal covering of a complex manifold is equipped with a uniform squeezing coordinate. Hence from Theorem 1, it is pseudoconvex and hence a Stein manifold. Furthermore, it supports a complete Kähler-Einstein g_{KE} metric of negative scaler curvature. Since g_{KE} is invariant under biholomorphism and hence the deck transformations, it descends to N. Hence the canonical line bundle of N is ample and the manifold is of general type. We denote by $h^{p,q}_{(2),v}(M)$ the von Neumann dimension of the space of L^2-harmonic (p,q)-forms on M with respect to the Kähler-Einstein metric (cf. [At]). Corollary 1 implies that the von-Neumann dimension $h^{p,q}_{(2),v}(M) = 0$ for $p+q<n$ and $h^{n,0}_{(2,v)}(M) > 0$, which implies that the corresponding Euler-Poincaré characteristics $(-1)^n\chi_{L^2,v}(M) > 0$. From Atiyah's Covering Index Theorem, $\chi_{L^2,v}(M) = \chi(M/\Gamma)$. Hence $(-1)^n\chi(M/\Gamma) > 0$.

We may apply the same argument to the holomorphic line bundle $2K$ on M. We use K to denote by the canonical line bundle on N and M. First of all $h^0_{L^2,v}(M, 2K) > 0$ by the usual L^2-estimates as used in [Y2], noting that g_{KE} has strictly negative Ricci curvature. The same L^2-estimates implies that $h^i_{L^2,v}(M, 2K) = 0$ for $i > 0$. Atiyah's Covering Index Theorem implies that $\chi(N, 2K) > 0$. On the other hand, from Kodaira's Vanishing Theorem or L^2-estimates, we conclude that $h^i(N, 2K) = 0$ for $i > 0$. Hence $h^0(N, 2K) = \chi(N, 2K) > 0$. This concludes the proof of Theorem 3.

Proof of Corollary 3 From Theorem 1, we know that Ω is pseudoconvex. It was proved by Mok-Yau in [MY] that a complete Kähler-Einstein metric of negative sectional curvature exists and its volume form is bounded from below by $\frac{1}{d^2(-\log d)^2}$ with respect to the Euclidean coordinates. Corollary 3 follows from the proof of Theorem 2 as the Bergman kernel is shown to be equivalent to the Kähler-Einstein volume form. Note that both of them transforms under a coordinate change by the same Jacobian determinant as in the proof of Lemma 2.

Proof of Corollary 4 Stehlé has proved in [St] the result that a locally trivial holomorphic fiber space with hyperconvex fibers and Stein base is Stein. Corollary 3 follows immediately from Theorem 2f.

Remark There are many positive results to Serre's problem, including the result of Siu [Si] when the fibers have trivial first Betti number, the result of Mok [Mo] when the fibers are Riemann surfaces, and the results of Diederich and Fornaess

[DF]. In general the problem has negative solution due to counterexamples such as the one given by Skoda in [Sk].

Proof of Theorem 4 As explained in the last section, Bers Embedding gives rise to a uniform squeezing coordinate system. All the results of Theorem 4a-e follow from the earlier results of this paper under the sole assumption that a uniform squeezing coordinate system exists for $\mathcal{T}_{g,n}$, which is provided by Bers Embedding. Theorem 4f also follows from Corollary 2 if we accept that $\mathcal{M}_{g,n}$ is quasi-projective, which is known classically by the well-known work of Baily for $n = 0$ and Knudsen-Mumford for the case of $n \neq 0$ (cf. [KM]).

The exact formula for the Euler characteristic of M has already been obtained by Harer-Zagier [HZ]. We just remark that using Kähler hyperbolicity and Atiyah's Covering Theorem as in Theorem 3, we may prove that Euler-Poincaré characteristic of M_1 satisfies $(-1)^n\chi(M_1) > 0$ as well. The only minor difference is that M_1 is now non-compact. However, it follows from Theorem 2 that (M_1, g_{KE}) has finite geometry and hence the chopping argument of Cheeger-Gromov [CG] shows that one may exhaust M_1 by appropriate relatively compact sets so that the contribution from the boundary tends to 0 as one takes the limit on the exhaustion. It follows that $(-1)^n\chi(M_1) > 0$.

This concludes the proof of Theorem 4.

Remark It is known that $g_K, g_C, g_B, g_{KE}, g_T$ and g_M are quasi-isometric on $\mathcal{M}_{g,n}$, where g_T is the Teichmüller metric and g_M is a Kähler metric constructed in [Mc], as is shown in [Y3]. It is also proved in [Mc] that any order of derivatives of g_M is bounded as well. Combining with Theorem 2c and the proof there, the difference $\|\nabla^{g_1}_{X_1,\cdots X_N} R^{g_1} - \nabla^{g_2}_{X_1,\cdots X_N} R^{g_2}\|_{g_1}$ is bounded for any g_1, g_2 chosen among g_B, g_{KE} and g_M. Hence g_B, g_{KE} and g_M are all comparable up to any order of derivatives.

References

[At] Atiyah, M. F., Elliptic operators, discrete groups, and von Neumann algebras, Astérisque 32-33(1976), 43-72.

[Au] Aubin, T., Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[B] Borel, A., Compact Clifford-Klein forms of symmetric spaces, Topology 2 (1963), 111-122.

[CG] Cheeger, J., Gromov, M., On the characteristic numbers of complete manifolds of bounded curvature and finite volume, Differential geometry and complex analysis, 115-154, Springer, Berlin, 1985.

[De] Demailly, J.-P., Champs magnetiques et inegalites de Morse pour la d"-cohomologie. 35 (1985), no. 4, 189–229.

[DF] Diederich, K., Fornaess, J. E., Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions. Invent. Math. 39 (1977), no. 2, 129–141.

[Do] Donnelly, H., L^2 cohomology of the Bergman metric for weakly pseudoconvex domains. Illinois J. Math. 41 (1997), no. 1, 151–160.

[Ga] Gardiner, F., Teichmüller theory and quadratic differentials, Wiley Interscience, New York, 1987.

[Gr] Gromov, M., Kähler hyperbolicity and L^2-Hodge theory. J. Differential Geom. 33 (1991), no. 1, 263–292.

[HZ] Harer, J., Zagier, D. The Euler characteristic of the moduli space of curves. Invent. Math. 85 (1986), no. 3, 457–485.

[He] Helgason, S., Differential geometry, Lie groups and symmetric spaces, Academic Press 1978

[KM] Knudsen, F. and Mumford, D., The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". Math. Scand. 39 (1976), no. 1, 19–55.

[Ko] Kollár, J., Shafarevich maps and automorphic forms. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1995.

[Kr] Krantz, S., Function theory of several complex variables. Second edition. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992. xvi+557 pp

[Mc] McMullen, C. T., The moduli space of Riemann surfaces is Kähler hyperbolic, Ann. of Math. 151 (2000), 327–357.

[Mo] Mok, N., The Serre problem on Riemann surfaces. Math. Ann. 258 (1981/82), no. 2, 145–168.

[MY] Mok, N., Yau, S., Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions. The mathematical heritage of Henri Poincaré, Part 1 (Bloomington, Ind., 1980), 41–59, Proc. Sympos. Pure Math., 39, Amer. Math. Soc., Providence, RI, 1983.

[MZ] Mok, N., Zhong, J. Q., Compactifying complete Kahler-Einstein manifolds of finite topological type and bounded curvature. Ann. of Math. (2) 129 (1989), no. 3, 427–470.

[NT] Nadel, A., Tsuji, H., Compactification of complete Kahler manifolds of negative Ricci curvature. J. Diff. Geom. 28 (1988), no. 3, 503–512.

[Ro] Royden, H., The Ahlfors-Schwarz lemma in several complex variables, Comment. Math. Helv. 55 (1980), no. 4, 547–558.

[Se] Serre, J.-P., Quelques problmes globaux relatifs aux variétés de Stein, Colloque sur les fonctions de plusieurs variables, tenu á Bruxelles, 1953, pp. 57–68. Georges Thone, Liège; Masson & Cie, Paris, 1953.

[Si] Siu, Y.-T., Holomorphic fiber bundles whose fibers are bounded Stein domains with zero first Betti number. Math. Ann. 219 (1976), no. 2, 171–192.

[Sk] Skoda, H., Fibrés holomorphes base et à fibre de Stein, Invent. Math. 43 (1977), no. 2, 97–107.

[St] Stehlé, J.-L., Fonctions plurisousharmoniques et convexité holomorphe de certains fibrés analytiques. (French) C. R. Acad. Sci. Paris Sér. A 279 (1974), 235–238.

[W] Wolpert, S., Weil-Petersson perspectives, arXiv: math.DG/0502519.

[Y1] Yeung, S.-K., Compactification of Kahler manifolds with negative Ricci curvature. Invent. Math. 106 (1991), no. 1, 13–25.

[Y2] Yeung, S.-K., Bounded smooth strictly plurisubharmonic exhaustion functions on Teichmüller spaces, Math. Res. Letters, (10) 2003, 391-400.

[Y3] Yeung, S.-K., Quasi-isometry of metrics on Teichmüller spaces, Int. Math. Res. Not. 2005, 239-255.

[Y4] Yeung, S.-K., Bergman metric on Teichmüller spaces and moduli spaces of curves, in Recent progress on some problems in several complex variables and partial differential equations, Contempory Mathematics 400.

Mathematics Department, Purdue University, West Lafayette, IN 47907 USA
E-mail address: `yeung@math.purdue.edu`